✔ QUICK CHECK EXERCISES 10.1 (See page 705 for answers.)

1. Find parametric equations for a circle of radius 2, centered at (3, 5).
2. The graph of the curve described by the parametric equations $x = 4t - 1$, $y = 3t + 2$ is a straight line with slope ______ and y-intercept ______.
3. Suppose that a parametric curve C is given by the equations $x = f(t)$, $y = g(t)$ for $0 \le t \le 1$. Find parametric equations for C that reverse the direction the curve is traced as the parameter increases from 0 to 1.
4. To find dy/dx directly from the parametric equation

$$x = f(t), \quad y = g(t)$$

we can use the formula $dy/dx =$ ______.
5. Let L be the length of the curve

$$x = \ln t, \quad y = \sin t \qquad (1 \le t \le \pi)$$

An integral expression for L is ______.

EXERCISE SET 10.1 Graphing Utility C CAS

1. (a) By eliminating the parameter, sketch the trajectory over the time interval $0 \le t \le 5$ of the particle whose parametric equations of motion are

$$x = t - 1, \quad y = t + 1$$

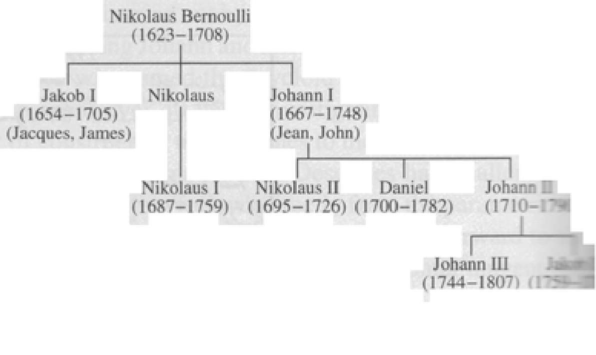
(b) Indicate the direction of motion on your sketch.
(c) Make a table of x- and y-coordinates of the par times $t = 0, 1, 2, 3, 4, 5$.
(d) Mark the position of the particle on the curve at the in part (c), and label those positions with the valu

Johann (left) **and Jakob** (right) **Bernoulli** Members of an amazing Swiss family that included several generations of outstanding mathematicians and scientists. Nikolaus Bernoulli (1623–1708), a druggist, fled from Antwerp to escape religious persecution and ultimately settled in Basel, Switzerland. There he had three sons, Jakob I (also called Jacques or James), Nikolaus, and Johann I (also called Jean or John). The Roman numerals are used to distinguish family members with identical names (see the family tree below). Following Newton and Leibniz, the Bernoulli brothers, Jakob I and Johann I, are considered by some to be the two most important founders of calculus. Jakob I was self-taught in mathematics. His father wanted him to study for the ministry, but he turned to mathematics and in 1686 became a professor at the University of Basel. When he started working in mathematics, he knew nothing of Newton's and Leibniz' work. He eventually became familiar with Newton's results, but because so little of Leibniz' work was published, Jakob duplicated many of Leibniz' results.

Jakob's younger brother Johann I was urged to enter into business by his father. Instead, he turned to medicine and studied mathematics under the guidance of his older brother. He eventually became a mathematics professor at Gröningen in Holland, and then, when Jakob died in 1705, Johann succeeded him as mathematics professor at Basel. Throughout their lives, Jakob I and Johann I had a mutual passion for criticizing each other's work, which frequently erupted into ugly confrontations. Leibniz tried to mediate the disputes, but Jakob, who resented Leibniz' superior intellect, accused him of siding with Johann, and thus Leibniz became entangled in the arguments. The brothers often worked on common problems that they posed as challenges to one another. Johann, interested in gaining fame, often used unscrupulous means to make himself appear the originator of his brother's results; Jakob occasionally retaliated. Thus, it is often difficult to determine who deserves credit for many results. However, both men made major contributions to the development of calculus. In addition to his work on c lus, Jakob helped establish fundamental principles in proba including the Law of Large Numbers, which is a cornersto modern probability theory.

Among the other members of the Bernoulli family, Danie of Johann I, is the most famous. He was a professor of mathem at St. Petersburg Academy in Russia and subsequently a prof of anatomy and then physics at Basel. He did work in calculu probability, but is best known for his work in physics. A basi of fluid flow, called Bernoulli's principle, is named in his hono won the annual prize of the French Academy 10 times for wor vibrating strings, tides of the sea, and kinetic theory of gases.

Johann II succeeded his father as professor of mathem Basel. His research was on the theory of heat and sound. Nik I was a mathematician and law scholar who worked on prob and series. On the recommendation of Leibniz, he was appo professor of mathematics at Padua and then went to Basel professor of logic and then law. Nikolaus II was professor ri risprudence in Switzerland and then professor of mathematics Petersburg Academy. Johann III was a professor of mathem and astronomy in Berlin and Jakob II succeeded his uncle D as professor of mathematics at St. Petersburg Academy in R Truly an incredible family!



a) By eliminating the parameter, sketch the trajectory over the time interval $0 \le t \le 1$ of the particle whose parametric equations of motion are

$$x = \cos(\pi t), \quad y = \sin(\pi t)$$

b) Indicate the direction of motion on your sketch.
c) Make a table of x- and y-coordinates of the particle at times $t = 0, 0.25, 0.5, 0.75, 1$.
d) Mark the position of the particle on the curve at the times in part (c), and label those positions with the values of t.

Sketch the curve by eliminating the parameter, and indi-
he direction of increasing t. ■

$= 3t - 4,\ y = 6t + 2$

$= t - 3,\ y = 3t - 7 \quad (0 \le t \le 3)$

$= 2\cos t,\ y = 5\sin t \quad (0 \le t \le 2\pi)$

$= \sqrt{t},\ y = 2t + 4$

$= 3 + 2\cos t,\ y = 2 + 4\sin t \quad (0 \le t \le 2\pi)$

$= \sec t,\ y = \tan t \quad (\pi \le t < 3\pi/2)$

$= \cos 2t,\ y = \sin t \quad (-\pi/2 \le t \le \pi/2)$

$= 4t + 3,\ y = 16t^2 - 9$

$= 2\sin^2 t,\ y = 3\cos^2 t \quad (0 \le t \le \pi/2)$

$= \sec^2 t,\ y = \tan^2 t \quad (0 \le t < \pi/2)$

Find parametric equations for the curve, and check your
by generating the curve with a graphing utility. ■

circle of radius 5, centered at the origin, oriented clock-
wise.

The portion of the circle $x^2 + y^2 = 1$ that lies in the third quadrant, oriented counterclockwise.

vertical line intersecting the x-axis at $x = 2$, oriented upward.

The ellipse $x^2/4 + y^2/9 = 1$, oriented counterclockwise.

The portion of the parabola $x = y^2$ joining $(1, -1)$ and $(1, 1)$, oriented down to up.

The circle of radius 4, centered at $(1, -3)$, oriented counterclockwise.

a) Use a graphing utility to generate the trajectory of a particle whose equations of motion over the time interval $0 \le t \le 5$ are

$$x = 6t - \tfrac{1}{2}t^3, \quad y = 1 + \tfrac{1}{2}t^2$$

b) Make a table of x- and y-coordinates of the particle at times $t = 0, 1, 2, 3, 4, 5$.
c) At what times is the particle on the y-axis?
d) During what time interval is $y < 5$?
e) At what time does the x-coordinate of the particle reach a maximum?

a) Use a graphing utility to generate the trajectory of a paper airplane whose equations of motion for $t \ge 0$ are

$$x = t - 2\sin t, \quad y = 3 - 2\cos t$$

(b) Assuming that the plane flies in a room in which the floor is at $y = 0$, explain why the plane will not crash into the floor. [For simplicity, ignore the physical size of the plane by treating it as a particle.]

(c) How high must the ceiling be to ensure that the plane does not touch or crash into it?

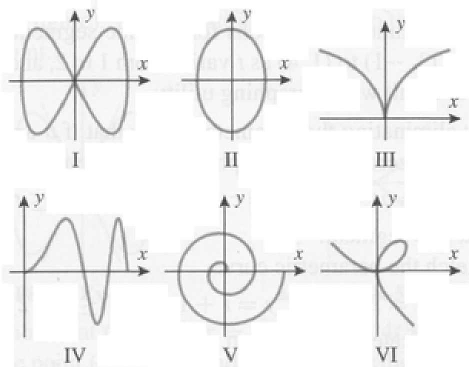
21–22 Graph the equation using a graphing utility. ■

21. (a) $x = y^2 + 2y + 1$
(b) $x = \sin y,\ -2\pi \le y \le 2\pi$

22. (a) $x = y + 2y^3 - y^5$
(b) $x = \tan y,\ -\pi/2 < y < \pi/2$

FOCUS ON CONCEPTS

23. In each part, match the parametric equation with one of the curves labeled (I)–(VI), and explain your reasoning.
(a) $x = \sqrt{t},\ y = \sin 3t$ (b) $x = 2\cos t,\ y = 3\sin t$
(c) $x = t\cos t,\ y = t\sin t$
(d) $x = \dfrac{3t}{1+t^3},\ y = \dfrac{3t^2}{1+t^3}$
(e) $x = \dfrac{t^3}{1+t^2},\ y = \dfrac{2t^2}{1+t^2}$
(f) $x = \frac{1}{2}\cos t,\ y = \sin 2t$



▲ **Figure Ex-23**

24. (a) Identify the orientation of the curves in Exercise 23.
(b) Explain why the parametric curve

$$x = t^2, \quad y = t^4 \quad (-1 \le t \le 1)$$

does not have a definite orientation.

25. (a) Suppose that the line segment from the point $P(x_0, y_0)$ to $Q(x_1, y_1)$ is represented parametrically by

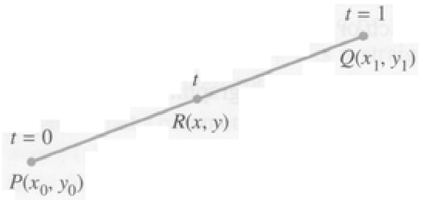
$$x = x_0 + (x_1 - x_0)t, \quad y = y_0 + (y_1 - y_0)t \quad (0 \le t \le 1)$$

and that $R(x, y)$ is the point on the line segment corresponding to a specified value of t (see the accompanying figure on the next page). Show that $t = r/q$, where r is the distance from P to R and q is the distance from P to Q.

(cont.)

(b) What value of t produces the midpoint between points P and Q?

(c) What value of t produces the point that is three-fourths of the way from P to Q?



◀ **Figure Ex-25**

26. Find parametric equations for the line segment joining $P(2, -1)$ and $Q(3, 1)$, and use the result in Exercise 25 to find

(a) the midpoint between P and Q

(b) the point that is one-fourth of the way from P to Q

(c) the point that is three-fourths of the way from P to Q.

27. (a) Show that the line segment joining the points (x_0, y_0) and (x_1, y_1) can be represented parametrically as

$$x = x_0 + (x_1 - x_0)\frac{t - t_0}{t_1 - t_0}, \qquad (t_0 \le t \le t_1)$$
$$y = y_0 + (y_1 - y_0)\frac{t - t_0}{t_1 - t_0}$$

(b) Which way is the line segment oriented?

(c) Find parametric equations for the line segment traced from $(3, -1)$ to $(1, 4)$ as t varies from 1 to 2, and check your result with a graphing utility.

28. (a) By eliminating the parameter, show that if a and c are not both zero, then the graph of the parametric equations

$$x = at + b, \quad y = ct + d \qquad (t_0 \le t \le t_1)$$

is a line segment.

(b) Sketch the parametric curve

$$x = 2t - 1, \quad y = t + 1 \qquad (1 \le t \le 2)$$

and indicate its orientation.

(c) What can you say about the line in part (a) if a or c (but not both) is zero?

(d) What do the equations represent if a and c are both zero?

29–32 Use a graphing utility and parametric equations to display the graphs of f and f^{-1} on the same screen. ■

29. $f(x) = x^3 + 0.2x - 1, \quad -1 \le x \le 2$

30. $f(x) = \sqrt{x^2 + 2} + x, \quad -5 \le x \le 5$

31. $f(x) = \cos(\cos 0.5x), \quad 0 \le x \le 3$

32. $f(x) = x + \sin x, \quad 0 \le x \le 6$

33–36 True–False Determine whether the statement is true or false. Explain your answer. ■

33. The equation $y = 1 - x^2$ can be described parametrically by $x = \sin t$, $y = \cos^2 t$.

34. The graph of the parametric equations $x = f(t)$, $y = t$ is the reflection of the graph of $y = f(x)$ about the x-axis.

35. For the parametric curve $x = x(t)$, $y = 3t^4 - 2t^3$, the deriv-
ative of y with respect to x is computed by

$$\frac{dy}{dx} = \frac{12t^3 - 6t^2}{x'(t)}$$

36. The curve represented by the parametric equations

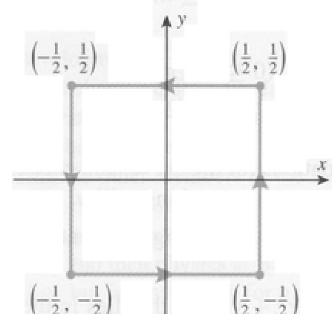
$$x = t^3, \quad y = t + t^6 \qquad (-\infty < t < +\infty)$$

is concave down for $t < 0$.

37. Parametric curves can be defined piecewise by using differ-
ent formulas for different values of the parameter. Sketch
the curve that is represented piecewise by the parametric
equations

$$\begin{cases} x = 2t, & y = 4t^2 \quad (0 \le t \le \frac{1}{2}) \\ x = 2 - 2t, & y = 2t \quad (\frac{1}{2} \le t \le \end{cases}$$

38. Find parametric equations for the rectangle in the accom-
panying figure, assuming that the rectangle is traced coun-
terclockwise as t varies from 0 to 1, starting at $(\frac{1}{2}, \frac{1}{2})$ when
$t = 0$. [*Hint:* Represent the rectangle piecewise, letting t
vary from 0 to $\frac{1}{4}$ for the first edge, from $\frac{1}{4}$ to $\frac{1}{2}$ for the second
edge, and so forth.]



◀ **Figure Ex-38**

39. (a) Find parametric equations for the ellipse that is centered
at the origin and has intercepts $(4, 0)$, $(-4, 0)$,
and $(0, -3)$.

(b) Find parametric equations for the ellipse that results from
translating the ellipse in part (a) so that its center is at
$(-1, 2)$.

(c) Confirm your results in parts (a) and (b) using a graph-
ing utility.

40. We will show later in the text that if a projectile is fired from
ground level with an initial speed of v_0 meters per second
at an angle α with the horizontal, and if air resistance is
neglected, then its position after t seconds, relative to the
coordinate system in the accompanying figure on the next
page is

$$x = (v_0 \cos\alpha)t, \quad y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2$$

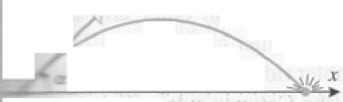
where $g \approx 9.8 \text{ m/s}^2$.

(a) By eliminating the parameter, show that the trajectory
lies on the graph of a quadratic polynomial.

(b) Use a graphing utility to sketch the trajectory if
and $v_0 = 1000$ m/s.

(c) Using the trajectory in part (b), how high does the
rise?

Using the trajectory in part (b), how far does the shell travel horizontally?



◀ **Figure Ex-40**

FOCUS ON CONCEPTS

(a) Find the slope of the tangent line to the parametric curve $x = t/2$, $y = t^2 + 1$ at $t = -1$ and at $t = 1$ without eliminating the parameter.
(b) Check your answers in part (a) by eliminating the parameter and differentiating an appropriate function of x.

(a) Find the slope of the tangent line to the parametric curve $x = 3\cos t$, $y = 4\sin t$ at $t = \pi/4$ and at $t = 7\pi/4$ without eliminating the parameter.
(b) Check your answers in part (a) by eliminating the parameter and differentiating an appropriate function of x.

For the parametric curve in Exercise 41, make a conjecture about the sign of d^2y/dx^2 at $t = -1$ and at $t = 1$, and confirm your conjecture without eliminating the parameter.

For the parametric curve in Exercise 42, make a conjecture about the sign of d^2y/dx^2 at $t = \pi/4$ and at $t = 7\pi/4$, and confirm your conjecture without eliminating the parameter.

Find dy/dx and d^2y/dx^2 at the given point without eliminating the parameter.

$x = \sqrt{t}$, $y = 2t + 4$; $t = 1$

$x = \frac{1}{2}t^2 + 1$, $y = \frac{1}{3}t^3 - t$; $t = 2$

$x = \sec t$, $y = \tan t$; $t = \pi/3$

$x = \sinh t$, $y = \cosh t$; $t = 0$

$x = \theta + \cos\theta$, $y = 1 + \sin\theta$; $\theta = \pi/6$

$x = \cos\phi$, $y = 3\sin\phi$; $\phi = 5\pi/6$

(a) Find the equation of the tangent line to the curve

$$x = e^t, \quad y = e^{-t}$$

at $t = 1$ without eliminating the parameter.
(b) Find the equation of the tangent line in part (a) by eliminating the parameter.

(a) Find the equation of the tangent line to the curve

$$x = 2t + 4, \quad y = 8t^2 - 2t + 4$$

at $t = 1$ without eliminating the parameter.
(b) Find the equation of the tangent line in part (a) by eliminating the parameter.

53–54 Find all values of t at which the parametric curve has (a) a horizontal tangent line and (b) a vertical tangent line. ■

53. $x = 2\sin t$, $y = 4\cos t$ $(0 \le t \le 2\pi)$

54. $x = 2t^3 - 15t^2 + 24t + 7$, $y = t^2 + t + 1$

55. In the mid-1850s the French physicist Jules Antoine Lissajous (1822–1880) became interested in parametric equations of the form

$$x = \sin at, \quad y = \sin bt$$

in the course of studying vibrations that combine two perpendicular sinusoidal motions. If a/b is a rational number, then the combined effect of the oscillations is a periodic motion along a path called a ***Lissajous curve***.
(a) Use a graphing utility to generate the complete graph of the Lissajous curves corresponding to $a = 1, b = 2$; $a = 2, b = 3$; $a = 3, b = 4$; and $a = 4, b = 5$.
(b) The Lissajous curve

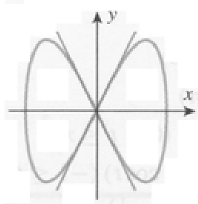
$$x = \sin t, \quad y = \sin 2t \qquad (0 \le t \le 2\pi)$$

crosses itself at the origin (see Figure Ex-55). Find equations for the two tangent lines at the origin.

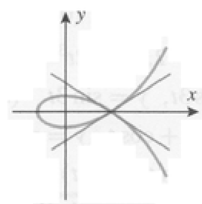
56. The ***prolate cycloid***

$$x = 2 - \pi\cos t, \quad y = 2t - \pi\sin t \qquad (-\pi \le t \le \pi)$$

crosses itself at a point on the x-axis (see the accompanying figure). Find equations for the two tangent lines at that point.



▲ **Figure Ex-55**



▲ **Figure Ex-56**

57. Show that the curve $x = t^2$, $y = t^3 - 4t$ intersects itself at the point $(4, 0)$, and find equations for the two tangent lines to the curve at the point of intersection.

58. Show that the curve with parametric equations

$$x = t^2 - 3t + 5, \quad y = t^3 + t^2 - 10t + 9$$

intersects itself at the point $(3, 1)$, and find equations for the two tangent lines to the curve at the point of intersection.

59. (a) Use a graphing utility to generate the graph of the parametric curve

$$x = \cos^3 t, \quad y = \sin^3 t \qquad (0 \le t \le 2\pi)$$

and make a conjecture about the values of t at which singular points occur.
(b) Confirm your conjecture in part (a) by calculating appropriate derivatives.

60. Verify that the cycloid described by Formula (10) has cusps at its x-intercepts and horizontal tangent lines at midpoints between adjacent x-intercepts (see Figure 10.1.14).

61. (a) What is the slope of the tangent line at time t to the trajectory of the paper airplane in Example 5?
(b) What was the airplane's approximate angle of inclination when it crashed into the wall?

62. Suppose that a bee follows the trajectory

$$x = t - 2\cos t, \quad y = 2 - 2\sin t \qquad (0 \le t \le 10)$$

(a) At what times was the bee flying horizontally?
(b) At what times was the bee flying vertically?

63. Consider the family of curves described by the parametric equations

$$x = a\cos t + h, \quad y = b\sin t + k \qquad (0 \le t < 2\pi)$$

where $a \neq 0$ and $b \neq 0$. Describe the curves in this family if
(a) h and k are fixed but a and b can vary
(b) a and b are fixed but h and k can vary
(c) $a = 1$ and $b = 1$, but h and k vary so that $h = k + 1$.

64. (a) Use a graphing utility to study how the curves in the family

$$x = 2a\cos^2 t, \quad y = 2a\cos t\sin t \qquad (-2\pi < t < 2\pi)$$

change as a varies from 0 to 5.
(b) Confirm your conclusion algebraically.
(c) Write a brief paragraph that describes your findings.

65–70 Find the exact arc length of the curve over the stated interval. ■

65. $x = t^2,\ y = \frac{1}{3}t^3 \quad (0 \le t \le 1)$

66. $x = \sqrt{t} - 2,\ y = 2t^{3/4} \quad (1 \le t \le 16)$

67. $x = \cos 3t,\ y = \sin 3t \quad (0 \le t \le \pi)$

68. $x = \sin t + \cos t,\ y = \sin t - \cos t \quad (0 \le t \le \pi)$

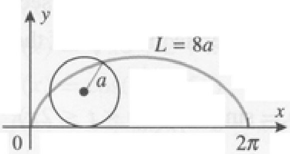
69. $x = e^{2t}(\sin t + \cos t),\ y = e^{2t}(\sin t - \cos t)\ (-1 \le t \le 1)$

70. $x = 2\sin^{-1} t,\ y = \ln(1 - t^2) \quad (0 \le t \le \frac{1}{2})$

C **71.** (a) Use Formula (9) to show that the length L of one arch of a cycloid is given by

$$L = a\int_0^{2\pi} \sqrt{2(1 - \cos\theta)}\, d\theta$$

(b) Use a CAS to show that L is eight times the radius of the wheel that generates the cycloid (see the accompanying figure).



◀ **Figure Ex-71**

72. Use the parametric equations in Formula (10) to verify that the cycloid provides one solution to the differential equation

$$\left(1 + \left(\frac{dy}{dx}\right)^2\right) y = 2a$$

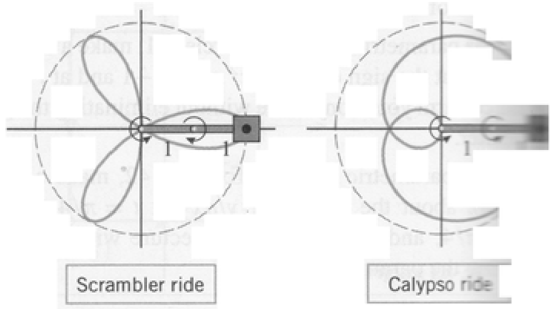
where a is a positive constant.

FOCUS ON CONCEPTS

73. The amusement park rides illustrated in the acco[...] nying figure consist of two connected rotating arm[...] length 1—an inner arm that rotates countercloc[...] at 1 radian per second and an outer arm that can[...] programmed to rotate either clockwise at 2 radian[...] second (the Scrambler ride) or counterclockwise at 2 [...] dians per second (the Calypso ride). The center of [...] rider cage is at the end of the outer arm.
(a) Show that in the Scrambler ride the center of [...] cage has parametric equations

$$x = \cos t + \cos 2t, \quad y = \sin t - \sin 2[\ldots]$$

(b) Find parametric equations for the center of the [...] in the Calypso ride, and use a graphing utili[...] confirm that the center traces the curve show[...] the accompanying figure.
(c) Do you think that a rider travels the same dist[...] in one revolution of the Scrambler ride as in [...] revolution of the Calypso ride? Justify your c[...] clusion.



▲ **Figure Ex-73**

74. (a) If a thread is unwound from a fixed circle whi[...] ing held taut (i.e., tangent to the circle), the [...] end of the thread traces a curve called an invol[...] ***of a circle***. Show that if the circle is center[...] the origin, has radius a, and the end of the thre[...] initially at the point $(a, 0)$, then the involute ca[...] expressed parametrically as

$$x = a(\cos\theta + \theta\sin\theta), \quad y = a(\sin\theta - [\ldots]$$

where θ is the angle shown in part (*a*) of the acc[...] panying figure on the next page.
(b) Assuming that the dog in part (*b*) of the accompa[...] ing figure on the next page unwinds its leash wh[...] keeping it taut, for what values of θ in the inter[...] $0 \le \theta \le 2\pi$ will the dog be walking North? So[...] East? West?
(c) Use a graphing utility to generate the curve tra[...] by the dog, and show that it is consistent with y[...] answer in part (*b*).

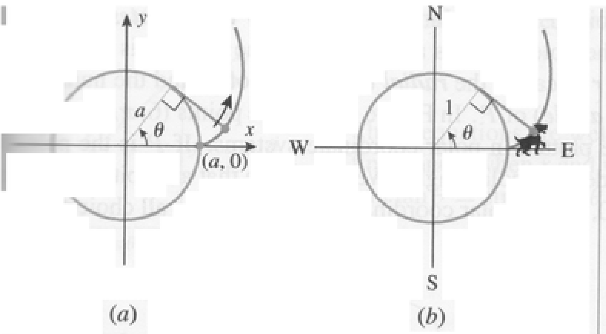


Figure Ex-74

If $f'(t)$ and $g'(t)$ are continuous functions, and if no ...ment of the curve

$$x = f(t), \quad y = g(t) \qquad (a \le t \le b)$$

...ced more than once, then it can be shown that the area of ...rface generated by revolving this curve about the x-axis is

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

...he area of the surface generated by revolving the curve ...t the y-axis is

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

...erivations are similar to those used to obtain Formulas ...and (5) in Section 6.5.] Use the formulas above in these ...cises.

75. Find the area of the surface generated by revolving $x = t^2$, $y = 3t$ $(0 \le t \le 2)$ about the x-axis.

76. Find the area of the surface generated by revolving the curve $x = e^t \cos t$, $y = e^t \sin t$ $(0 \le t \le \pi/2)$ about the x-axis.

77. Find the area of the surface generated by revolving the curve $x = \cos^2 t$, $y = \sin^2 t$ $(0 \le t \le \pi/2)$ about the y-axis.

78. Find the area of the surface generated by revolving $x = 6t$, $y = 4t^2$ $(0 \le t \le 1)$ about the y-axis.

79. By revolving the semicircle

$$x = r\cos t, \quad y = r\sin t \qquad (0 \le t \le \pi)$$

about the x-axis, show that the surface area of a sphere of radius r is $4\pi r^2$.

80. The equations

$$x = a\phi - a\sin\phi, \quad y = a - a\cos\phi \qquad (0 \le \phi \le 2\pi)$$

represent one arch of a cycloid. Show that the surface area generated by revolving this curve about the x-axis is given by $S = 64\pi a^2/3$.

81. **Writing** Consult appropriate reference works and write an essay on American mathematician Nathaniel Bowditch (1773–1838) and his investigation of ***Bowditch curves*** (better known as Lissajous curves; see Exercise 55).

82. **Writing** What are some of the advantages of expressing a curve parametrically rather than in the form $y = f(x)$?

QUICK CHECK ANSWERS 10.1

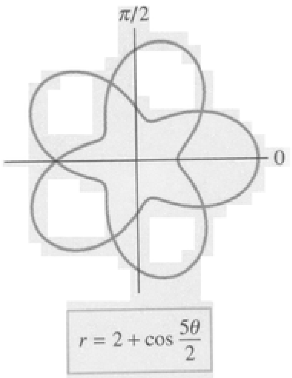
$= 3 + 2\cos t,\ y = 5 + 2\sin t \quad (0 \le t \le 2\pi)$ 2. $\frac{3}{4}$; 2.75 3. $x = f(1-t),\ y = g(1-t)$ 4. $\dfrac{dy/dt}{dx/dt} = \dfrac{g'(t)}{f'(t)}$
$\sqrt{(1/t)^2 + \cos^2 t}\, dt$

POLAR COORDINATES

Up to now we have specified the location of a point in the plane by means of coordinates relative to two perpendicular coordinate axes. However, sometimes a moving point has a special affinity for some fixed point, such as a planet moving in an orbit under the central attraction of the Sun. In such cases, the path of the particle is best described by its angular direction and its distance from the fixed point. In this section we will discuss a new kind of coordinate system that is based on this idea.

POLAR COORDINATE SYSTEMS

A ***polar coordinate system*** in a plane consists of a fixed point O, called the ***pole*** (or ***origin***), and a ray emanating from the pole, called the ***polar axis***. In such a coordinate system



▲ Figure 10.2.23

For this equality to hold, the quantity $5n\pi$ must be an even multiple of π; the small[est] … which this occurs is $n = 2$. Thus, the entire graph will be traced in two revolution[s] … means it can be generated from the parametric equations

$$x = \left[2 + \cos\frac{5\theta}{2}\right]\cos\theta, \quad y = \left[2 + \cos\frac{5\theta}{2}\right]\sin\theta \qquad (0 \le \theta \le 4\pi)$$

This yields the graph in Figure 10.2.23. ◄

✔QUICK CHECK EXERCISES 10.2 *(See page 719 for answers.)*

1. (a) Rectangular coordinates of a point (x, y) may be recovered from its polar coordinates (r, θ) by means of the equations $x =$ ______ and $y =$ ______.
   (b) Polar coordinates (r, θ) may be recovered from rectangular coordinates (x, y) by means of the equations $r^2 =$ ______ and $\tan\theta =$ ______.

2. Find the rectangular coordinates of the points whose polar coordinates are given.
   (a) $(4, \pi/3)$ (b) $(2, -\pi/6)$
   (c) $(6, -2\pi/3)$ (d) $(4, 5\pi/4)$

3. In each part, find polar coordinates satisfying [the stated] conditions for the point whose rectangular coo[rdinates are] $(1, \sqrt{3})$.
   (a) $r \ge 0$ and $0 \le \theta < 2\pi$
   (b) $r \le 0$ and $0 \le \theta < 2\pi$

4. In each part, state the name that describes the p[olar curve] most precisely: a rose, a line, a circle, a limaçon, … a spiral, a lemniscate, or none of these.
   (a) $r = 1 - \theta$ (b) $r = 1 + 2\sin$…
   (c) $r = \sin 2\theta$ (d) $r = \cos^2\theta$
   (e) $r = \csc\theta$ (f) $r = 2 + 2\cos$…
   (g) $r = -2\sin\theta$

EXERCISE SET 10.2 Graphing Utility

1–2 Plot the points in polar coordinates. ■

1. (a) $(3, \pi/4)$ (b) $(5, 2\pi/3)$ (c) $(1, \pi/2)$
   (d) $(4, 7\pi/6)$ (e) $(-6, -\pi)$ (f) $(-1, 9\pi/4)$

2. (a) $(2, -\pi/3)$ (b) $(3/2, -7\pi/4)$ (c) $(-3, 3\pi/2)$
   (d) $(-5, -\pi/6)$ (e) $(2, 4\pi/3)$ (f) $(0, \pi)$

3–4 Find the rectangular coordinates of the points whose polar coordinates are given. ■

3. (a) $(6, \pi/6)$ (b) $(7, 2\pi/3)$ (c) $(-6, -5\pi/6)$
   (d) $(0, -\pi)$ (e) $(7, 17\pi/6)$ (f) $(-5, 0)$

4. (a) $(-2, \pi/4)$ (b) $(6, -\pi/4)$ (c) $(4, 9\pi/4)$
   (d) $(3, 0)$ (e) $(-4, -3\pi/2)$ (f) $(0, 3\pi)$

5. In each part, a point is given in rectangular coordinates. Find two pairs of polar coordinates for the point, one pair satisfying $r \ge 0$ and $0 \le \theta < 2\pi$, and the second pair satisfying $r \ge 0$ and $-2\pi < \theta \le 0$.
   (a) $(-5, 0)$ (b) $(2\sqrt{3}, -2)$ (c) $(0, -2)$
   (d) $(-8, -8)$ (e) $(-3, 3\sqrt{3})$ (f) $(1, 1)$

6. In each part, find polar coordinates satisfying the stated conditions for the point whose rectangular coordinates are $(-\sqrt{3}, 1)$.
   (a) $r \ge 0$ and $0 \le \theta < 2\pi$
   (b) $r \le 0$ and $0 \le \theta < 2\pi$
   (c) $r \ge 0$ and $-2\pi < \theta \le 0$
   (d) $r \le 0$ and $-\pi < \theta \le \pi$

7–8 Use a calculating utility, where needed, to appr[oximate the] polar coordinates of the points whose rectangular c[oordinates] are given. ■

7. (a) $(3, 4)$ (b) $(6, -8)$ (c) (–…
8. (a) $(-3, 4)$ (b) $(-3, 1.7)$ (c) (2, …

9–10 Identify the curve by transforming the given p[olar equa]tion to rectangular coordinates. ■

9. (a) $r = 2$ (b) $r\sin\theta = 4$
   (c) $r = 3\cos\theta$ (d) $r = \frac{6}{3\cos\theta + \ldots}$

10. (a) $r = 5\sec\theta$ (b) $r = 2\sin\theta$
    (c) $r = 4\cos\theta + 4\sin\theta$ (d) $r = \sec\theta\tan$…

11–12 Express the given equations in polar coordi[nates].

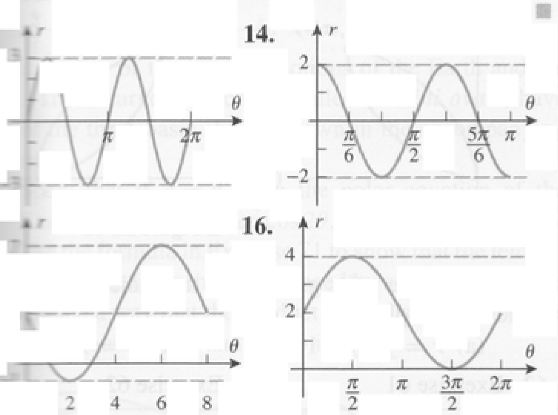
11. (a) $x = 3$ (b) $x^2 + y^2 = \frac{\ldots}{7}$
    (c) $x^2 + y^2 + 6y = 0$ (d) $9xy = 4$

$y = -3$ (b) $x^2 + y^2 = 5$

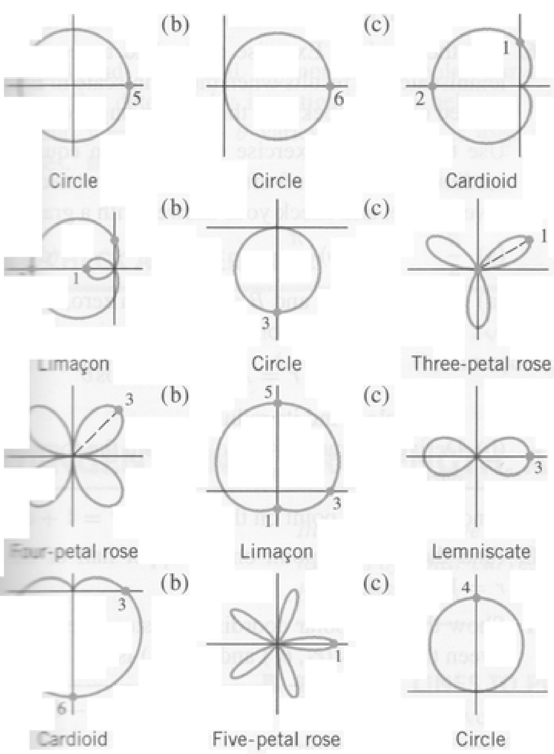
$x^2 + y^2 + 4x = 0$ (d) $x^2(x^2 + y^2) = y^2$

…S ON CONCEPTS

…–16 A graph is given in a rectangular θr-coordinate system. Sketch the corresponding graph in polar coordinates.



… Find an equation for the given polar graph. [*Note:* Numeric labels on these graphs represent distances to the origin.]



… Sketch the curve in polar coordinates.

22. $\theta = -\dfrac{3\pi}{4}$ **23.** $r = 3$

25. $r = 6\sin\theta$ **26.** $r - 2 = 2\cos\theta$

27. $r = 3(1 + \sin\theta)$ **28.** $r = 5 - 5\sin\theta$

29. $r = 4 - 4\cos\theta$ **30.** $r = 1 + 2\sin\theta$

31. $r = -1 - \cos\theta$ **32.** $r = 4 + 3\cos\theta$

33. $r = 3 - \sin\theta$ **34.** $r = 3 + 4\cos\theta$

35. $r - 5 = 3\sin\theta$ **36.** $r = 5 - 2\cos\theta$

37. $r = -3 - 4\sin\theta$ **38.** $r^2 = \cos 2\theta$

39. $r^2 = 16\sin 2\theta$ **40.** $r = 4\theta \quad (\theta \geq 0)$

41. $r = 4\theta \quad (\theta \leq 0)$ **42.** $r = 4\theta$

43. $r = -2\cos 2\theta$ **44.** $r = 3\sin 2\theta$

45. $r = 9\sin 4\theta$ **46.** $r = 2\cos 3\theta$

47–50 True–False Determine whether the statement is true or false. Explain your answer.

47. The polar coordinate pairs $(-1, \pi/3)$ and $(1, -2\pi/3)$ describe the same point.

48. If the graph of $r = f(\theta)$ drawn in rectangular θr-coordinates is symmetric about the r-axis, then the graph of $r = f(\theta)$ drawn in polar coordinates is symmetric about the x-axis.

49. The portion of the polar graph of $r = \sin 2\theta$ for values of θ between $\pi/2$ and π is contained in the second quadrant.

50. The graph of a dimpled limaçon passes through the polar origin.

51–55 Determine a shortest parameter interval on which a complete graph of the polar equation can be generated, and then use a graphing utility to generate the polar graph.

51. $r = \cos\dfrac{\theta}{2}$ **52.** $r = \sin\dfrac{\theta}{2}$

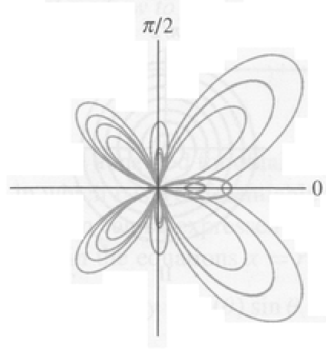
53. $r = 1 - 2\sin\dfrac{\theta}{4}$ **54.** $r = 0.5 + \cos\dfrac{\theta}{3}$

55. $r = \cos\dfrac{\theta}{5}$

56. The accompanying figure shows the graph of the "butterfly curve"

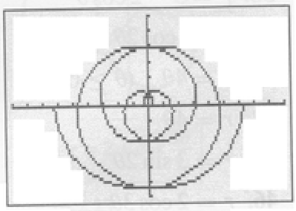
$$r = e^{\cos\theta} - 2\cos 4\theta + \sin^3\frac{\theta}{4}$$

Determine a shortest parameter interval on which the complete butterfly can be generated, and then check your answer using a graphing utility.



◀ **Figure Ex-56**

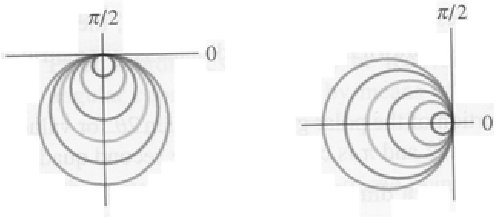
57. The accompanying figure shows the Archimedean spiral $r = \theta/2$ produced with a graphing calculator.
(a) What interval of values for θ do you think was used to generate the graph?
(b) Duplicate the graph with your own graphing utility.



$[-9, 9] \times [-6, 6]$
$x\text{Scl} = 1, y\text{Scl} = 1$

◀ **Figure Ex-57**

58. Find equations for the two families of circles in the accompanying figure.

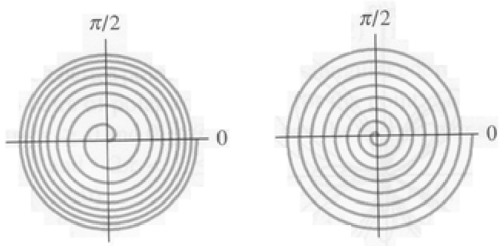


▲ **Figure Ex-58**

59. (a) Show that if a varies, then the polar equation
$$r = a \sec\theta \quad (-\pi/2 < \theta < \pi/2)$$
describes a family of lines perpendicular to the polar axis.
(b) Show that if b varies, then the polar equation
$$r = b \csc\theta \quad (0 < \theta < \pi)$$
describes a family of lines parallel to the polar axis.

FOCUS ON CONCEPTS

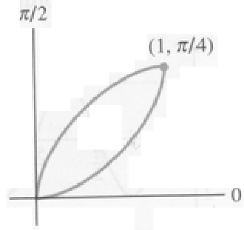
60. The accompanying figure shows graphs of the Archimedean spiral $r = \theta$ and the parabolic spiral $r = \sqrt{\theta}$. Which is which? Explain your reasoning.



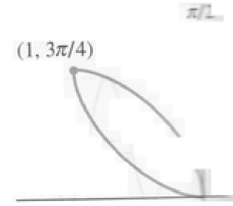
▲ **Figure Ex-60**

61–62 A polar graph of $r = f(\theta)$ is given over the sta[...] interval. Sketch the graph of
(a) $r = f(-\theta)$
(b) $r = f\left(\theta - \frac{\pi}{2}\right)$
(c) $r = f\left(\theta + \frac{\pi}{2}\right)$
(d) $r = -f(\theta)$

61. $0 \le \theta \le \pi/2$
62. $\pi/2 \le \theta \le \pi$



▲ **Figure Ex-61**



▲ **Figure Ex-62**

63–64 Use the polar graph from the indicated exercise [...] sketch the graph of
(a) $r = f(\theta) + 1$
(b) $r = 2f(\theta - [...]$

63. Exercise 61
64. Exercise 62

65. Show that if the polar graph of $r = f(\theta)$ is rotated [...]terclockwise around the origin through an angle [...] $r = f(\theta - \alpha)$ is an equation for the rotated curve. [...] (r_0, θ_0) is any point on the original graph, then [...] is a point on the rotated graph.]

66. Use the result in Exercise 65 to find an equation [...] lemniscate that results when the lemniscate in Ex[...] rotated counterclockwise through an angle of $\pi/2$.

67. Use the result in Exercise 65 to find an equation [...] cardioid $r = 1 + \cos\theta$ after it has been rotated [...] given angle, and check your answer with a graphing [...]
(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) π (d) [...]

68. (a) Show that if A and B are not both zero, then the [...] of the polar equation
$$r = A\sin\theta + B\cos\theta$$
is a circle. Find its radius.
(b) Derive Formulas (4) and (5) from the formula [...] part (a).

69. Find the highest point on the cardioid $r = 1 + \cos\theta$[...]

70. Find the leftmost point on the upper half of the [...] $r = 1 + \cos\theta$.

71. Show that in a polar coordinate system the distance [...]tween the points (r_1, θ_1) and (r_2, θ_2) is
$$d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}$$

72–74 Use the formula obtained in Exercise 71 to find [...] tance between the two points indicated in polar coordinates.

72. $(3, \pi/6)$ and $(2, \pi/3)$

Successive tips of the four-petal rose $r = \cos 2\theta$. Check your answer using geometry.

Successive tips of the three-petal rose $r = \sin 3\theta$. Check your answer using trigonometry.

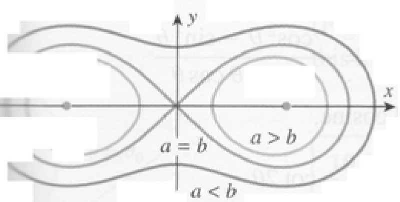
In the late seventeenth century the Italian astronomer Giovanni Domenico Cassini (1625–1712) introduced the family of curves

$$(x^2 + y^2 + a^2)^2 - b^4 - 4a^2x^2 = 0 \quad (a > 0, b > 0)$$

in his studies of the relative motions of the Earth and the Sun. These curves, which are called ***Cassini ovals***, have one of the three basic shapes shown in the accompanying figure.

(a) Show that if $a = b$, then the polar equation of the Cassini oval is $r^2 = 2a^2 \cos 2\theta$, which is a lemniscate.

(b) Use the formula in Exercise 71 to show that the lemniscate in part (a) is the curve traced by a point that moves in such a way that the product of its distances from the polar points $(a, 0)$ and (a, π) is a^2.



▲ Figure Ex-75

Vertical and horizontal asymptotes of polar curves can sometimes be detected by investigating the behavior of $x = r\cos\theta$ and $y = r \sin\theta$ as θ varies. This idea is used in the exercises.

76. Show that the ***hyperbolic spiral*** $r = 1/\theta$ $(\theta > 0)$ has a horizontal asymptote at $y = 1$ by showing that $y \to 1$ and $x \to +\infty$ as $\theta \to 0^+$. Confirm this result by generating the spiral with a graphing utility.

77. Show that the spiral $r = 1/\theta^2$ does not have any horizontal asymptotes.

78. Prove that a rose with an even number of petals is traced out exactly once as θ varies over the interval $0 \le \theta < 2\pi$ and a rose with an odd number of petals is traced out exactly once as θ varies over the interval $0 \le \theta < \pi$.

79. (a) Use a graphing utility to confirm that the graph of $r = 2 - \sin(\theta/2)$ $(0 \le \theta \le 4\pi)$ is symmetric about the x-axis.

(b) Show that replacing θ by $-\theta$ in the polar equation $r = 2 - \sin(\theta/2)$ does not produce an equivalent equation. Why does this not contradict the symmetry demonstrated in part (a)?

80. Writing Use a graphing utility to investigate how the family of polar curves $r = 1 + a\cos n\theta$ is affected by changing the values of a and n, where a is a positive real number and n is a positive integer. Write a brief paragraph to explain your conclusions.

81. Writing Why do you think the adjective "polar" was chosen in the name "polar coordinates"?

QUICK CHECK ANSWERS 10.2

$r\cos\theta$; $r\sin\theta$ (b) $x^2 + y^2$; y/x 2. (a) $(2, 2\sqrt{3})$ (b) $(\sqrt{3}, -1)$ (c) $(-3, -3\sqrt{3})$ (d) $(-2\sqrt{2}, -2\sqrt{2})$ ... $-\pi/3)$ (b) $(-2, 4\pi/3)$ 4. (a) spiral (b) limaçon (c) rose (d) none of these (e) line (f) cardioid (g) circle

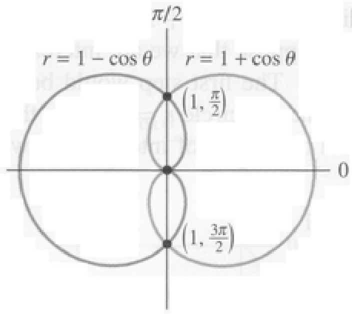
TANGENT LINES, ARC LENGTH, AND AREA FOR POLAR CURVES

In this section we will derive the formulas required to find slopes, tangent lines, and arc lengths of polar curves. We will then show how to find areas of regions that are bounded by polar curves.

TANGENT LINES TO POLAR CURVES

Our first objective in this section is to find a method for obtaining slopes of tangent lines to polar curves of the form $r = f(\theta)$ in which r is a differentiable function of θ. We showed in the last section that a curve of this form can be expressed parametrically in terms of the parameter θ by substituting $f(\theta)$ for r in the equations $x = r\cos\theta$ and $y = r\sin\theta$. This yields

$$x = f(\theta)\cos\theta, \quad y = f(\theta)\sin\theta$$



▲ Figure 10.3.13

INTERSECTIONS OF POLAR GRAPHS

In the last example we found the intersections of the cardioid and circle by equating t expressions for r and solving for θ. However, because a point can be represented in diffe ways in polar coordinates, this procedure will not always produce all of the intersect For example, the cardioids

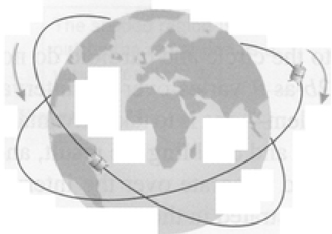
$$r = 1 - \cos\theta \quad \text{and} \quad r = 1 + \cos\theta$$

intersect at three points: the pole, the point $(1, \pi/2)$, and the point $(1, 3\pi/2)$ (Fig 10.3.13). Equating the right-hand sides of the equations in (7) yields $1 - \cos\theta =$ 1 − c or $\cos\theta = 0$, so

$$\theta = \frac{\pi}{2} + k\pi, \quad k = 0, \pm 1, \pm 2, \ldots$$

Substituting any of these values in (7) yields $r = 1$, so that we have found only two dis points of intersection, $(1, \pi/2)$ and $(1, 3\pi/2)$; the pole has been missed. This pro occurs because the two cardioids pass through the pole at different values of θ—the card $r = 1 - \cos\theta$ passes through the pole at $\theta = 0$, and the cardioid $r = 1 + \cos\theta$ pa through the pole at $\theta = \pi$.

The situation with the cardioids is analogous to two satellites circling the Earth in in secting orbits (Figure 10.3.14). The satellites will not collide unless they reach the s point at the same time. In general, when looking for intersections of polar curves, it good idea to graph the curves to determine how many intersections there should be


The orbits intersect, but the satellites do not collide.

▲ Figure 10.3.14

✔ QUICK CHECK EXERCISES 10.3

(See page 729 for answers.)

1. (a) To obtain dy/dx directly from the polar equation $r = f(\theta)$, we can use the formula

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \_\_\_\_\_\_\_\_$$

(b) Use the formula in part (a) to find dy/dx directly from the polar equation $r = \csc\theta$.

2. (a) What conditions on $f(\theta_0)$ and $f'(\theta_0)$ guarantee that the line $\theta = \theta_0$ is tangent to the polar curve $r = f(\theta)$ at the origin?

(b) What are the values of θ_0 in $[0, 2\pi]$ at which the $\theta = \theta_0$ are tangent at the origin to the four-peta $r = \cos 2\theta$?

3. (a) To find the arc length L of the polar curve $r =$ $(\alpha \le \theta \le \beta)$, we can use the formula $L = \_\_\_\_$

(b) The polar curve $r = \sec\theta$ $(0 \le \theta \le \pi/4)$ has arc $L = \_\_\_\_$.

4. The area of the region enclosed by a nonnegative po $r = f(\theta)$ $(\alpha \le \theta \le \beta)$ and the lines $\theta = \alpha$ and $\theta =$ given by the definite integral ________.

5. Find the area of the circle $r = a$ by integration

EXERCISE SET 10.3

Graphing Utility C CAS

1–6 Find the slope of the tangent line to the polar curve for the given value of θ.

1. $r = 2\sin\theta;\ \theta = \pi/6$

2. $r = 1 + \cos\theta;\ \theta = \pi/2$

3. $r = 1/\theta;\ \theta = 2$

4. $r = a\sec 2\theta;\ \theta = \pi/6$

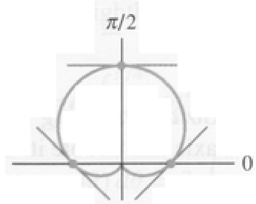
5. $r = \sin 3\theta;\ \theta = \pi/4$

6. $r = 4 - 3\sin\theta;\ \theta = \pi$

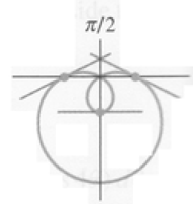
7–8 Calculate the slopes of the tangent lines indicated in the accompanying figures.

7. $r = 2 + 2\sin\theta$

8. $r = 1 - 2\sin\theta$



▲ Figure Ex-7



▲ Figure Ex-8

9–10 Find polar coordinates of all points at which the curve has a horizontal or a vertical tangent line.

9. $r = a(1 + \cos\theta)$

10. $r = a\sin\theta$

2/ Use a graphing utility to make a conjecture about the ...ber of points on the polar curve at which there is a horizontal ...ent line, and confirm your conjecture by finding appropriate ...atives.

$r = \sin\theta\cos^2\theta$ **12.** $r = 1 - 2\sin\theta$

...8 Sketch the polar curve and find polar equations of the ...ent lines to the curve at the pole.

$r = 2\cos 3\theta$ **14.** $r = 4\sin\theta$ **15.** $r = 4\sqrt{\cos 2\theta}$

$r = \sin 2\theta$ **17.** $r = 1 - 2\cos\theta$ **18.** $r = 2\theta$

...22 Use Formula (3) to calculate the arc length of the polar ...e

The entire circle $r = a$

The entire circle $r = 2a\cos\theta$

The entire cardioid $r = a(1 - \cos\theta)$

$r = e^{3\theta}$ from $\theta = 0$ to $\theta = 2$

(a) Show that the arc length of one petal of the rose $r = \cos n\theta$ is given by

$$2\int_0^{\pi/(2n)} \sqrt{1 + (n^2 - 1)\sin^2 n\theta}\, d\theta$$

(b) Use the numerical integration capability of a calculating utility to approximate the arc length of one petal of the four-petal rose $r = \cos 2\theta$.

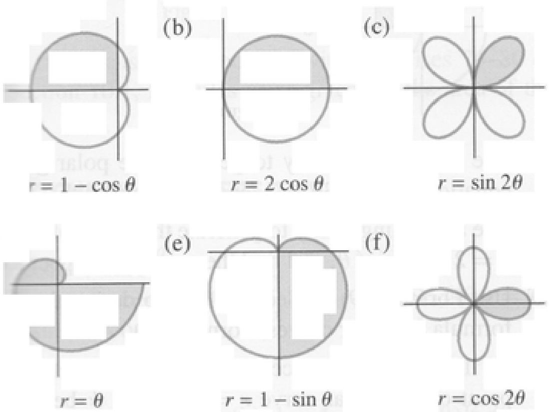
(c) Use the numerical integration capability of a calculating utility to approximate the arc length of one petal of the n-petal rose $r = \cos n\theta$ for $n = 2, 3, 4, \ldots, 20$; then make a conjecture about the limit of these arc lengths as $n \to +\infty$.

(a) Sketch the spiral $r = e^{-\theta/8}$ $(0 \le \theta < +\infty)$.

(b) Find an improper integral for the total arc length of the spiral.

(c) Show that the integral converges and find the total arc length of the spiral.

Write down, but do not evaluate, an integral for the area of each shaded region.



Find the area of the shaded region in Exercise 25(d).

27. In each part, find the area of the circle by integration.
(a) $r = 2a\sin\theta$ (b) $r = 2a\cos\theta$

28. (a) Show that $r = 2\sin\theta + 2\cos\theta$ is a circle.
(b) Find the area of the circle using a geometric formula and then by integration.

29–34 Find the area of the region described.

29. The region that is enclosed by the cardioid $r = 2 + 2\sin\theta$.

30. The region in the first quadrant within the cardioid $r = 1 + \cos\theta$.

31. The region enclosed by the rose $r = 4\cos 3\theta$.

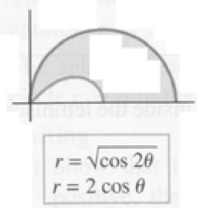
32. The region enclosed by the rose $r = 2\sin 2\theta$.

33. The region enclosed by the inner loop of the limaçon $r = 1 + 2\cos\theta$. [*Hint:* $r \le 0$ over the interval of integration.]

34. The region swept out by a radial line from the pole to the curve $r = 2/\theta$ as θ varies over the interval $1 \le \theta \le 3$.

35–38 Find the area of the shaded region.

35.



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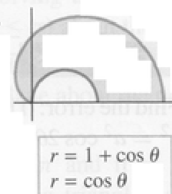

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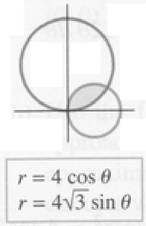

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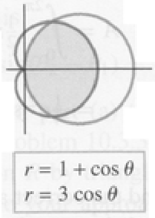


39–46 Find the area of the region described.

39. The region inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.

40. The region outside the cardioid $r = 2 - 2\cos\theta$ and inside the circle $r = 4$.

41. The region inside the cardioid $r = 2 + 2\cos\theta$ and outside the circle $r = 3$.

42. The region that is common to the circles $r = 2\cos\theta$ and $r = 2\sin\theta$.

43. The region between the loops of the limaçon $r = \frac{1}{2} + \cos\theta$.

44. The region inside the cardioid $r = 2 + 2\cos\theta$ and to the right of the line $r\cos\theta = \frac{3}{2}$.

45. The region inside the circle $r = 2$ and to the right of the line $r = \sqrt{2}\sec\theta$.

46. The region inside the rose $r = 2a\cos 2\theta$ and outside the circle $r = a\sqrt{2}$.

47–50 True–False Determine whether the statement is true or false. Explain your answer. ■

47. The x-axis is tangent to the polar curve $r = \cos(\theta/2)$ at $\theta = 3\pi$.

48. The arc length of the polar curve $r = \sqrt{\theta}$ for $0 \le \theta \le \pi/2$ is given by

$$L = \int_0^{\pi/2} \sqrt{1 + \frac{1}{4\theta}}\, d\theta$$

49. The area of a sector with central angle θ taken from a circle of radius r is θr^2.

50. The expression

$$\frac{1}{2}\int_{-\pi/4}^{\pi/4} (1 - \sqrt{2}\cos\theta)^2\, d\theta$$

computes the area enclosed by the inner loop of the limaçon $r = 1 - \sqrt{2}\cos\theta$.

FOCUS ON CONCEPTS

51. (a) Find the error: The area that is inside the lemniscate $r^2 = a^2 \cos 2\theta$ is

$$A = \int_0^{2\pi} \tfrac{1}{2} r^2\, d\theta = \int_0^{2\pi} \tfrac{1}{2} a^2 \cos 2\theta\, d\theta$$
$$= \tfrac{1}{4} a^2 \sin 2\theta \Big]_0^{2\pi} = 0$$

(b) Find the correct area.

(c) Find the area inside the lemniscate $r^2 = 4\cos 2\theta$ and outside the circle $r = \sqrt{2}$.

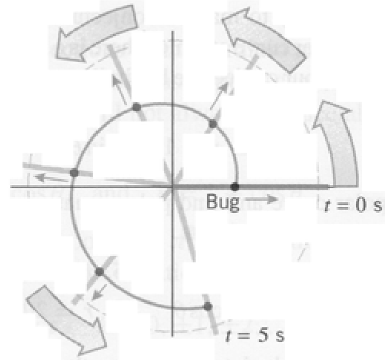
52. Find the area inside the curve $r^2 = \sin 2\theta$.

53. A radial line is drawn from the origin to the spiral $r = a\theta$ ($a > 0$ and $\theta \ge 0$). Find the area swept out during the second revolution of the radial line that was not swept out during the first revolution.

54. As illustrated in the accompanying figure, suppose that a rod with one end fixed at the pole of a polar coordinate system rotates counterclockwise at the constant rate of 1 rad/s. At time $t = 0$ a bug on the rod is 10 mm from the pole and is moving outward along the rod at the constant speed of 2 mm/s.

(a) Find an equation of the form $r = f(\theta)$ for the path of motion of the bug, assuming that $\theta = 0$ when $t = 0$.

(b) Find the distance the bug travels along the path in part (a) during the first 5 s. Round your answer to the nearest tenth of a millimeter.



◀ Figure Ex-54

C 55. (a) Show that the Folium of Descartes $x^3 - 3xy + \ldots$ can be expressed in polar coordinates as

$$r = \frac{3\sin\theta\cos\theta}{\cos^3\theta + \sin^3\theta}$$

(b) Use a CAS to show that the area inside of the ... (Figure 3.1.3a).

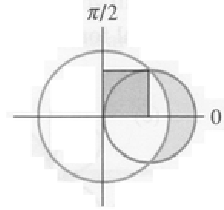
C 56. (a) What is the area that is enclosed by one petal ... $r = a\cos n\theta$ if n is an even integer?

(b) What is the area that is enclosed by one petal ... $r = a\cos n\theta$ if n is an odd integer?

(c) Use a CAS to show that the total area enclosed ... rose $r = a\cos n\theta$ is $\pi a^2/2$ if the number of ... even. [*Hint:* See Exercise 78 of Section 10.2.]

(d) Use a CAS to show that the total area enclosed ... rose $r = a\cos n\theta$ is $\pi a^2/4$ if the number of ... odd.

57. One of the most famous problems in Greek antiquity ... "squaring the circle," that is, using a straightedge and ... pass to construct a square whose area is equal to ... given circle. It was proved in the nineteenth century ... no such construction is possible. However, show ... shaded areas in the accompanying figure are equal ... "squaring the crescent."



◀ Figure Ex-57

58. Use a graphing utility to generate the polar graph ... equation $r = \cos 3\theta + 2$, and find the area that it ...

59. Use a graphing utility to generate the graph of the ... $r = 2\cos\theta\sin^2\theta$, and find the area of the upper ...

60. Use Formula (9) of Section 10.1 to derive the ... formula for polar curves, Formula (3).

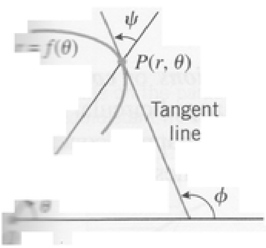
61. As illustrated in the accompanying figure, let P ... point on the polar curve $r = f(\theta)$, let ψ be the ... counterclockwise angle from the extended radius ...

tangent line at P, and let ϕ be the angle of inclination of the tangent line. Derive the formula

$$\tan\psi = \frac{r}{dr/d\theta}$$

by substituting $\tan\phi$ for dy/dx in Formula (2) and applying the trigonometric identity

$$\tan(\phi - \theta) = \frac{\tan\phi - \tan\theta}{1 + \tan\phi\tan\theta}$$



◀ **Figure Ex-61**

Use the formula for ψ obtained in Exercise 61. ■

(a) Use the trigonometric identity

$$\tan\frac{\theta}{2} = \frac{1 - \cos\theta}{\sin\theta}$$

to show that if (r, θ) is a point on the cardioid

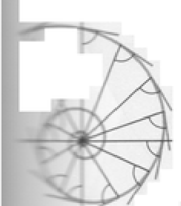
$$r = 1 - \cos\theta \qquad (0 \le \theta < 2\pi)$$

then $\psi = \theta/2$.

(b) Sketch the cardioid and show the angle ψ at the points where the cardioid crosses the y-axis.

(c) Find the angle ψ at the points where the cardioid crosses the y-axis.

Show that for a logarithmic spiral $r = ae^{b\theta}$, the angle from the radial line to the tangent line is constant along the spiral (see the accompanying figure). [*Note:* For this reason, logarithmic spirals are sometimes called ***equiangular spirals***.]



◀ **Figure Ex-63**

In the discussion associated with Exercises 75–80 of Section 10.1, formulas were given for the area of the surface of revolution that is generated by revolving a parametric curve about the x-axis or y-axis. Use those formulas to derive the following formulas for the areas of the surfaces of revolution that are generated by revolving the portion of the polar curve $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ about the polar axis and about the line $\theta = \pi/2$:

$$S = \int_\alpha^\beta 2\pi r\sin\theta\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta \qquad \text{About } \theta = 0$$

$$S = \int_\alpha^\beta 2\pi r\cos\theta\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta \qquad \text{About } \theta = \pi/2$$

(b) State conditions under which these formulas hold.

65–68 Sketch the surface, and use the formulas in Exercise 64 to find the surface area. ■

65. The surface generated by revolving the circle $r = \cos\theta$ about the line $\theta = \pi/2$.

66. The surface generated by revolving the spiral $r = e^\theta$ $(0 \le \theta \le \pi/2)$ about the line $\theta = \pi/2$.

67. The "apple" generated by revolving the upper half of the cardioid $r = 1 - \cos\theta$ $(0 \le \theta \le \pi)$ about the polar axis.

68. The sphere of radius a generated by revolving the semicircle $r = a$ in the upper half-plane about the polar axis.

69. Writing

(a) Show that if $0 \le \theta_1 < \theta_2 \le \pi$ and if r_1 and r_2 are positive, then the area A of a triangle with vertices $(0, 0)$, (r_1, θ_1), and (r_2, θ_2) is

$$A = \tfrac{1}{2}r_1 r_2\sin(\theta_2 - \theta_1)$$

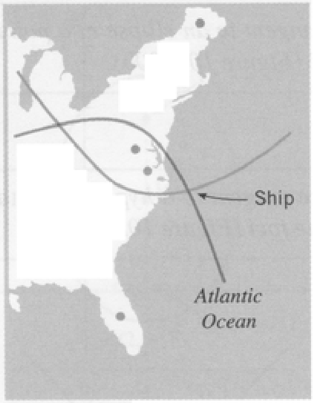
(b) Use the formula obtained in part (a) to describe an approach to answer Area Problem 10.3.3 that uses an approximation of the region R by triangles instead of circular wedges. Reconcile your approach with Formula (6).

70. Writing In order to find the area of a region bounded by two polar curves it is often necessary to determine their points of intersection. Give an example to illustrate that the points of intersection of curves $r = f(\theta)$ and $r = g(\theta)$ may not coincide with solutions to $f(\theta) = g(\theta)$. Discuss some strategies for determining intersection points of polar curves and provide examples to illustrate your strategies.

QUICK CHECK ANSWERS 10.3

$\dfrac{r\cos\theta + \sin\theta\dfrac{dr}{d\theta}}{-r\sin\theta + \cos\theta\dfrac{dr}{d\theta}}$ (b) $\dfrac{dy}{dx} = 0$ 2. (a) $f(\theta_0) = 0,\ f'(\theta_0) \ne 0$ (b) $\theta_0 = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

$\sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2}\,d\theta$ (b) 1 4. $\displaystyle\int_\alpha^\beta \tfrac{1}{2}[f(\theta)]^2\,d\theta = \int_\alpha^\beta \tfrac{1}{2}r^2\,d\theta$ 5. $\displaystyle\int_0^{2\pi} \tfrac{1}{2}a^2\,d\theta = \pi a^2$



▲ Figure 10.4.32

synchronized radio signals from two widely spaced transmitters with known positions. The ship's electronic receiver measures the difference in reception times between the signals and then uses that difference to compute the difference $2a$ between its distances from the two transmitters. This information places the ship somewhere on the hyperbola whose foci are at the transmitters and whose points have $2a$ as the difference in their distances from the foci. By repeating the process with a second set of transmitters, the position of the ship can be approximated as the intersection of two hyperbolas (Figure 10.4.32). (The modern global positioning system (GPS) is based on the same principle.)

✔ QUICK CHECK EXERCISES 10.4 (See page 748 for answers.)

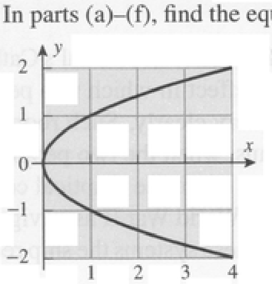
1. Identify the conic.
   (a) The set of points in the plane, the sum of whose distances to two fixed points is a positive constant greater than the distance between the fixed points is ____.
   (b) The set of points in the plane, the difference of whose distances to two fixed points is a positive constant less than the distance between the fixed points is ____.
   (c) The set of points in the plane that are equidistant from a fixed line and a fixed point not on the line is ____.
2. (a) The equation of the parabola with focus $(p, 0)$ and directrix $x = -p$ is ____.
   (b) The equation of the parabola with focus $(0, p)$ and directrix $y = -p$ is ____.
3. (a) Suppose that an ellipse has semimajor axis a and semiminor axis b. Then for all points on the ellipse, the sum of the distances to the foci is equal to ____.
   (b) The two standard equations of an ellipse with semimajor axis a and semiminor axis b are ____ and ____.
   (c) Suppose that an ellipse has semimajor axis a, semiminor axis b, and foci $(\pm c, 0)$. Then c may be obtained from a and b by the equation $c =$ ____.
4. (a) Suppose that a hyperbola has semifocal axis a and semiconjugate axis b. Then for all points on the hyperbola, the difference of the distance to the farther focus minus the distance to the closer focus is equal to ____.
   (b) The two standard equations of a hyperbola with semifocal axis a and semiconjugate axis b are ____ and ____.
   (c) Suppose that a hyperbola in standard position has semifocal axis a, semiconjugate axis b, and foci $(\pm c, 0)$. Then c may be obtained from a and b by the equation $c =$ ____. The equations of the asymptotes of the hyperbola are $y = \pm$ ____.

EXERCISE SET 10.4 Graphing Utility

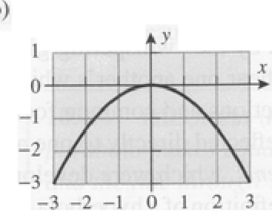
FOCUS ON CONCEPTS

1. In parts (a)–(f), find the equation of the conic.

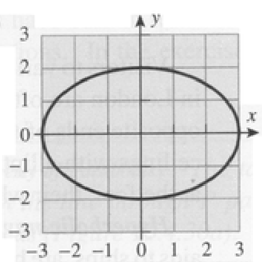
(a)



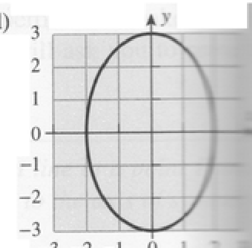
(b)

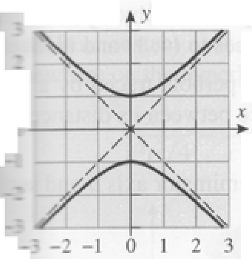


(c)

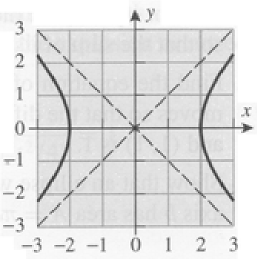


(d)





(f)



Find the focus and directrix for each parabola in Exercise 1.

(b) Find the foci of the ellipses in Exercise 1.

(c) Find the foci and the equations of the asymptotes of the hyperbolas in Exercise 1.

Sketch the parabola, and label the focus, vertex, and directrix.

(a) $y^2 = 4x$ (b) $x^2 = -8y$

(a) $y^2 = -10x$ (b) $x^2 = 4y$

(a) $(y-1)^2 = -12(x+4)$ (b) $(x-1)^2 = 2\left(y-\frac{1}{2}\right)$

(a) $y^2 - 6y - 2x + 1 = 0$ (b) $y = 4x^2 + 8x + 5$

Sketch the ellipse, and label the foci, vertices, and ends of the minor axis.

(a) $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$ (b) $9x^2 + y^2 = 9$

(a) $\dfrac{x^2}{25} + \dfrac{y^2}{4} = 1$ (b) $4x^2 + y^2 = 36$

(a) $(x+3)^2 + 4(y-5)^2 = 16$

(b) $\frac{1}{2}x^2 + \frac{1}{9}(y+2)^2 - 1 = 0$

(a) $9x^2 + 4y^2 - 18x + 24y + 9 = 0$

(b) $5x^2 + 9y^2 + 20x - 54y = -56$

Sketch the hyperbola, and label the vertices, foci, and asymptotes.

(a) $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$ (b) $9y^2 - x^2 = 36$

(a) $\dfrac{y^2}{9} - \dfrac{x^2}{25} = 1$ (b) $16x^2 - 25y^2 = 400$

(a) $\dfrac{(y+4)^2}{3} - \dfrac{(x-2)^2}{5} = 1$

(b) $16(x+1)^2 - 8(y-3)^2 = 16$

(a) $x^2 - 4y^2 + 2x + 8y - 7 = 0$

(b) $16x^2 - y^2 - 32x - 6y = 57$

Find an equation for the parabola that satisfies the given conditions.

(a) Vertex $(0, 0)$; focus $(3, 0)$.

(b) Vertex $(0, 0)$; directrix $y = \frac{1}{4}$.

(a) Focus $(6, 0)$; directrix $x = -6$.

(b) Focus $(1, 1)$; directrix $y = -2$.

Axis $y = 0$; passes through $(3, 2)$ and $(2, -\sqrt{2})$.

18. Vertex $(5, -3)$; axis parallel to the y-axis; passes through $(9, 5)$.

19–22 Find an equation for the ellipse that satisfies the given conditions. ■

19. (a) Ends of major axis $(\pm 3, 0)$; ends of minor axis $(0, \pm 2)$.
(b) Length of minor axis 8; foci $(0, \pm 3)$.

20. (a) Foci $(\pm 1, 0)$; $b = \sqrt{2}$.
(b) $c = 2\sqrt{3}$; $a = 4$; center at the origin; foci on a coordinate axis (two answers).

21. (a) Ends of major axis $(0, \pm 6)$; passes through $(-3, 2)$.
(b) Foci $(-1, 1)$ and $(-1, 3)$; minor axis of length 4.

22. (a) Center at $(0, 0)$; major and minor axes along the coordinate axes; passes through $(3, 2)$ and $(1, 6)$.
(b) Foci $(2, 1)$ and $(2, -3)$; major axis of length 6.

23–26 Find an equation for a hyperbola that satisfies the given conditions. [*Note:* In some cases there may be more than one hyperbola.] ■

23. (a) Vertices $(\pm 2, 0)$; foci $(\pm 3, 0)$.
(b) Vertices $(0, \pm 2)$; asymptotes $y = \pm\frac{2}{3}x$.

24. (a) Asymptotes $y = \pm\frac{3}{2}x$; $b = 4$.
(b) Foci $(0, \pm 5)$; asymptotes $y = \pm 2x$.

25. (a) Asymptotes $y = \pm\frac{3}{4}x$; $c = 5$.
(b) Foci $(\pm 3, 0)$; asymptotes $y = \pm 2x$.

26. (a) Vertices $(0, 6)$ and $(6, 6)$; foci 10 units apart.
(b) Asymptotes $y = x - 2$ and $y = -x + 4$; passes through the origin.

27–30 True–False Determine whether the statement is true or false. Explain your answer. ■

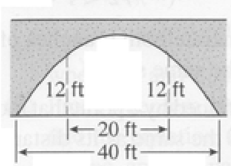
27. A hyperbola is the set of all points in the plane that are equidistant from a fixed line and a fixed point not on the line.

28. If an ellipse is not a circle, then the foci of an ellipse lie on the major axis of the ellipse.

29. If a parabola has equation $y^2 = 4px$, where p is a positive constant, then the perpendicular distance from the parabola's focus to its directrix is p.

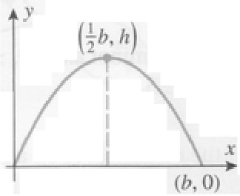
30. The hyperbola $(y^2/a^2) - x^2 = 1$ has asymptotes the lines $y = \pm x/a$.

31. (a) As illustrated in the accompanying figure, a parabolic arch spans a road 40 ft wide. How high is the arch if a center section of the road 20 ft wide has a minimum clearance of 12 ft?
(b) How high would the center be if the arch were the upper half of an ellipse?



◄ **Figure Ex-31**

32. (a) Find an equation for the parabolic arch with base b and height h, shown in the accompanying figure.
(b) Find the area under the arch.

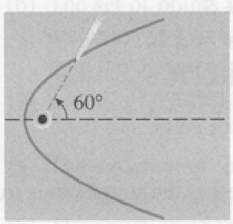


◄ **Figure Ex-32**

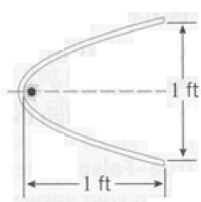
33. Show that the vertex is the closest point on a parabola to the focus. [*Suggestion:* Introduce a convenient coordinate system and use Definition 10.4.1.]

34. As illustrated in the accompanying figure, suppose that a comet moves in a parabolic orbit with the Sun at its focus and that the line from the Sun to the comet makes an angle of 60° with the axis of the parabola when the comet is 40 million miles from the center of the Sun. Use the result in Exercise 33 to determine how close the comet will come to the center of the Sun.

35. For the parabolic reflector in the accompanying figure, how far from the vertex should the light source be placed to produce a beam of parallel rays?



▲ **Figure Ex-34**



▲ **Figure Ex-35**

36. (a) Show that the right and left branches of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

can be represented parametrically as

$$x = a\cosh t, \quad y = b\sinh t \qquad (-\infty < t < +\infty)$$
$$x = -a\cosh t, \quad y = b\sinh t \qquad (-\infty < t < +\infty)$$

(b) Use a graphing utility to generate both branches of the hyperbola $x^2 - y^2 = 1$ on the same screen.

37. (a) Show that the right and left branches of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

can be represented parametrically as

$$x = a\sec t, \quad y = b\tan t \qquad (-\pi/2 < t < \pi/2)$$
$$x = -a\sec t, \quad y = b\tan t \qquad (-\pi/2 < t < \pi/2)$$

(b) Use a graphing utility to generate both branches of the hyperbola $x^2 - y^2 = 1$ on the same screen.

38. Find an equation of the parabola traced by a point that moves so that its distance from (2, 4) is the same as its distance to the x-axis.

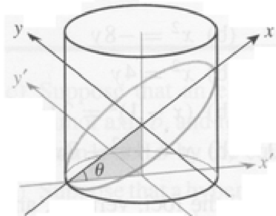
39. Find an equation of the ellipse traced by a point that moves so that the sum of its distances to (4, 1) and (4, 5

40. Find the equation of the hyperbola traced by a point that moves so that the difference between its distances to and (1, 1) is 1.

41. Show that an ellipse with semimajor axis a and semi axis b has area $A = \pi ab$.

FOCUS ON CONCEPTS

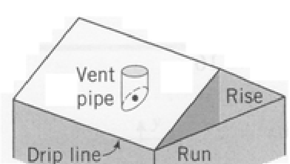
42. Show that if a plane is not parallel to the axis of a right circular cylinder, then the intersection of the plane and cylinder is an ellipse (possibly a circle). [*Hint:* Let the angle shown in the accompanying figure, introduce coordinate axes as shown, and express x' and y' in terms of x and y.]



◄ **Figure Ex-42**

43. As illustrated in the accompanying figure, a carpenter needs to cut an elliptical hole in a sloped roof through which a circular vent pipe of diameter D is to be inserted vertically. The carpenter wants to draw the outline of the hole on the roof using a pencil, two tacks, and a piece of string (as in Figure 10.4.3*b*). The center point of the ellipse is known, and common sense suggests that the major axis must be perpendicular to the drip line of the roof. The carpenter needs to determine the length L of the string and the distance T between a tack and the center point. The architect's plans show that the pitch of the roof is p (pitch = rise over run; see the accompanying figure). Find T and L in terms of D and p.

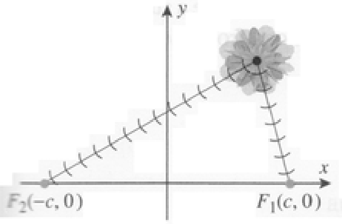
Source: This exercise is based on an article by William H. appeared in the *Mathematics Teacher*, Feb. 1991, p. 148.



◄ **Figure Ex-43**

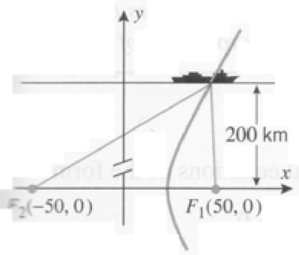
44. As illustrated in the accompanying figure on the page, suppose that two observers are stationed at points $F_1(c, 0)$ and $F_2(-c, 0)$ in an xy-coordinate system. Suppose also that the sound of an explosion in the xy-plane is heard by the F_1 observer t seconds before is heard by the F_2 observer. Assuming that the speed of sound is a constant v, show that the explosion occurred somewhere on the hyperbola

$$\frac{x^2}{v^2t^2/4} - \frac{y^2}{c^2 - (v^2t^2/4)} = 1$$



◀ **Figure Ex-44**

45. As illustrated in the accompanying figure, suppose that two transmitting stations are positioned 100 km apart at points $F_1(50, 0)$ and $F_2(-50, 0)$ on a straight shoreline in an xy-coordinate system. Suppose also that a ship is traveling parallel to the shoreline but 200 km at sea. Find the coordinates of the ship if the stations transmit a pulse simultaneously, but the pulse from station F_1 is received by the ship 100 microseconds sooner than the pulse from station F_2. [*Hint:* Use the formula obtained in Exercise 44, assuming that the pulses travel at the speed of light (299,792,458 m/s).]



◀ **Figure Ex-45**

46. A nuclear cooling tower is to have a height of h feet and the shape of the solid that is generated by revolving the region R enclosed by the right branch of the hyperbola $1521x^2 - 225y^2 = 342{,}225$ and the lines $x = 0$, $y = -h/2$, and $y = h/2$ about the y-axis.
(a) Find the volume of the tower.
(b) Find the lateral surface area of the tower.

47. Let R be the region that is above the x-axis and enclosed between the curve $b^2x^2 - a^2y^2 = a^2b^2$ and the line $x = \sqrt{a^2 + b^2}$.
(a) Sketch the solid generated by revolving R about the x-axis, and find its volume.
(b) Sketch the solid generated by revolving R about the y-axis, and find its volume.

48. Prove: The line tangent to the parabola $x^2 = 4py$ at the point (x_0, y_0) is $x_0x = 2p(y + y_0)$.

49. Prove: The line tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0) has the equation

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1$$

50. Prove: The line tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0) has the equation

$$\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1$$

51. Use the results in Exercises 49 and 50 to show that if an ellipse and a hyperbola have the same foci, then at each point of intersection their tangent lines are perpendicular.

52. Consider the second-degree equation

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

where A and C are not both 0. Show by completing the square:
(a) If $AC > 0$, then the equation represents an ellipse, a circle, a point, or has no graph.
(b) If $AC < 0$, then the equation represents a hyperbola or a pair of intersecting lines.
(c) If $AC = 0$, then the equation represents a parabola, a pair of parallel lines, or has no graph.

53. In each part, use the result in Exercise 52 to make a statement about the graph of the equation, and then check your conclusion by completing the square and identifying the graph.
(a) $x^2 - 5y^2 - 2x - 10y - 9 = 0$
(b) $x^2 - 3y^2 - 6y - 3 = 0$
(c) $4x^2 + 8y^2 + 16x + 16y + 20 = 0$
(d) $3x^2 + y^2 + 12x + 2y + 13 = 0$
(e) $x^2 + 8x + 2y + 14 = 0$
(f) $5x^2 + 40x + 2y + 94 = 0$

54. Derive the equation $x^2 = 4py$ in Figure 10.4.6.

55. Derive the equation $(x^2/b^2) + (y^2/a^2) = 1$ given in Figure 10.4.14.

56. Derive the equation $(x^2/a^2) - (y^2/b^2) = 1$ given in Figure 10.4.22.

57. Prove Theorem 10.4.4. [*Hint:* Choose coordinate axes so that the parabola has the equation $x^2 = 4py$. Show that the tangent line at $P(x_0, y_0)$ intersects the y-axis at $Q(0, -y_0)$ and that the triangle whose three vertices are at P, Q, and the focus is isosceles.]

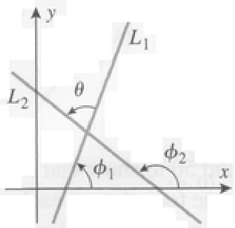
58. Given two intersecting lines, let L_2 be the line with the larger angle of inclination ϕ_2, and let L_1 be the line with the smaller angle of inclination ϕ_1. We define the ***angle θ between L_1 and L_2*** by $\theta = \phi_2 - \phi_1$. (See the accompanying figure on the next page.)
(a) Prove: If L_1 and L_2 are not perpendicular, then

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1m_2}$$

where L_1 and L_2 have slopes m_1 and m_2.
(b) Prove Theorem 10.4.5. [*Hint:* Introduce coordinates so that the equation $(x^2/a^2) + (y^2/b^2) = 1$ describes the ellipse, and use part (a).]

(cont.)

(c) Prove Theorem 10.4.6. [*Hint:* Introduce coordinates so that the equation $(x^2/a^2) - (y^2/b^2) = 1$ describes the hyperbola, and use part (a).]



◀ **Figure Ex-58**

59. Writing Suppose that you want to draw an ellipse ... given values for the lengths of the major and minor axes ... using the method shown in Figure 10.4.3*b*. Assuming ... the axes are drawn, explain how a compass can be ... locate the positions for the tacks.

60. Writing List the forms for standard equations of ... ellipses, and hyperbolas, and write a summary of ... for sketching conic sections from their standard ...

✔ QUICK CHECK ANSWERS 10.4

1. (a) an ellipse (b) a hyperbola (c) a parabola 2. (a) $y^2 = 4px$ (b) $x^2 = 4py$
3. (a) $2a$ (b) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$; $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$ (c) $\sqrt{a^2 - b^2}$ 4. (a) $2a$ (b) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$; $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$ (c) $\sqrt{a^2 + b^2}$

10.5 ROTATION OF AXES; SECOND-DEGREE EQUATIONS

In the preceding section we obtained equations of conic sections with axes parallel to the coordinate axes. In this section we will study the equations of conics that are ... relative to the coordinate axes. This will lead us to investigate rotations of coordinate axes.

■ QUADRATIC EQUATIONS IN *x* AND *y*

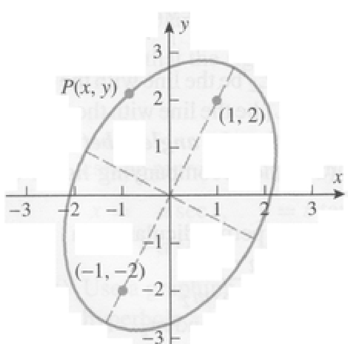
We saw in Examples 8 to 10 of the preceding section that equations of the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

can represent conic sections. Equation (1) is a special case of the more general equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

which, if A, B, and C are not all zero, is called a ***quadratic equation*** in x and ... usually the case that the graph of any second-degree equation is a conic section. If ... then (2) reduces to (1) and the conic section has its axis or axes parallel to the coordinate axes. However, if $B \neq 0$, then (2) contains a ***cross-product term*** Bxy, and the ... the conic section represented by the equation has its axis or axes "tilted" relative to the coordinate axes. As an illustration, consider the ellipse with foci $F_1(1, 2)$ and $F_2(-1, -2)$ and such that the sum of the distances from each point $P(x, y)$ on the ellipse to the foci is 6 units. Expressing this condition as an equation, we obtain (Figure 10.5.1)



▲ **Figure 10.5.1**

$$\sqrt{(x-1)^2 + (y-2)^2} + \sqrt{(x+1)^2 + (y+2)^2} = 6$$

Squaring both sides, then isolating the remaining radical, then squaring again ... yields

$$8x^2 - 4xy + 5y^2 = 36$$

as the equation of the ellipse. This is of form (2) with $A = 8$, $B = -4$, $C = 5$, ... $E = 0$, and $F = -36$.