SOME FUN WITH π

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Abstract. Today we will go over some fun activities for middle school and high school students related to *pi*. We will first introduce the Monte Carlo method as a means of computing probabilities by dropping a penny and a toothpick onto a sheet of lines. Then see how the Buffon's needle (the toothpick drop) is related to calculating */pi*. Since our accuracy of computing */pi* will be quite limited in the needle experiment we will look at a few infinite series to use to compute π.

1. Monte Carlo Method

We will start with a simple probability example. What is the probability of drawing a pair in a five card hand from a typical deck of playing cards?

We may recall some formula like

$$\frac{{}_4C_{2\,3}C_{48}}{{}_{52}C_5} = \frac{\frac{4!}{2!2!}\frac{13!}{1!12!}}{\frac{52!}{5!47!}} = \frac{1}{33320}.$$

Well maybe you do recall it or maybe you don't. And this doesn't eventake into account the possibility of two pairs or a full house. Either way there may be an easier way.

We could just play 10000 poker games and see how often I get a pair. Well maybe we can ask a computer for help to play all of those games. When we use such a simulation that is called a Monte Carlo Simulation (in reference to the gambling done at Monte Carlo). Some problems may be very easy to compute like the pair problem above. While other problems may be exceedingly difficult. Let's look at a few today.

1.1. **Drop a Penny Experiment.**

Question. Drop a penny onto the grid below (See Figure 3). What is the probability that the penny lands on a line.

Again the probability might be difficult to compute, but we can simulate this easily by dropping a few pennies. For this experiment drop 10, 20 30 and 50 pennies and record how many times they touch a line.

(1) What are your conclusions, thoughts and any suggested improvements?
(2) What do think the probability is from your data?

Penny Drop Data Sheet		
Count of Penny Drops	Count of Line Hits	Proportion $\frac{hits}{drops}$

Toothpick Drop Data Sheet			
Count of Toothpick Drops	Count of Line Hits	Proportion $\frac{hits}{drops}$	Formula $\frac{2}{Proportion}$
10			
20			
30			
40			
50			

(3) Try and compute the theoretical probability for the penny drop experiment.

1.2. **Drop a Toothpick Experiment.** Question. Drop a toothpick onto the same grid below (See Figure 3). What is the probability that the toothpick lands on a line.

Again we can simulate this easily by dropping a several toothpicks. For this experiment drop 10, 20 30 and 50 toothpicks and record how many times they touch a line. Additionally, we would like you to compute one additional calculation (th last column in the table below). Compute the fraction of 2 divided by the proportion.

(1) What are your conclusions, thoughts and any suggested improvements?

(2) What do think the probability is from your data? What did you get from your last column?
(3) Try and compute the theoretical probability for the toothpick drop experiment.

1.3. **The Theoretical proportions.**

1.3.1. *The Penny.* When I checked it was about 3.2 pennies between lines. If the penny touches the line the center of the penny is within 0.5 penny widths from the line. So if the penny center is 0.5 penny widths or closer to the top line or the penny center is 0.5 penny widths or closer to the bottom line then the penny will touch a line. Otherwise, the penny will not touch a line. So for a range of 2.2 penny widths the penny will not touch the line. The probability that a penny touches a line is

$$p = \frac{1}{3.2} \approx 31.25\%.$$

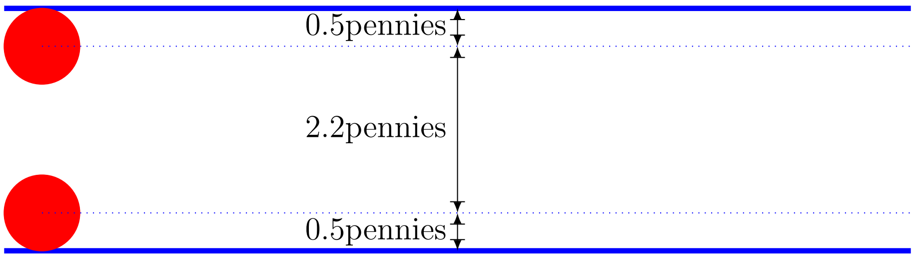

FIGURE 1. How does the penny fit

1.3.2. *The Toothpick.* The question about the toothpick is a more famous problem than the penny problem and was originally stated as a problem about a needle. The problem is called *Buffon's Needle.* It relates to an old gambling game posed in the 18th century by Georges-Louis Leclerc, Comte de Buffon.

We drop a needle onto a floor made of parallel slats of wood. The floor slats width are equal to the needle's length. What is the probability that the dropped needle will lie across a line between two slats?

Check out `https://mste.illinois.edu/activity/buffon/` for a cool simulation. Now to the proof.

First we will assume our needle is 1 unit long and that the distance from the bottom slat line to the top slat line is 1 unit also. We will let x be the distance from the bottom slat line to the center of the needle. Thus we have the perpendicular distance from highest point of the needle to the bottom slat is $x + h$ where h is the vertical in the triangle in Figure 2. Also let θ be represented as in Figure 2. So the ranges of our two variables x and θ are

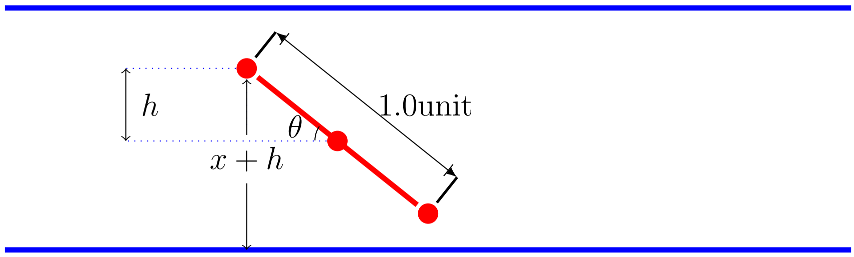

FIGURE 2. The needle dropped between two slat lines.

$$x \in [0, 1] \text{ and } \theta \in [0, \pi].$$

Note that the needle does **not** intersect a line exactly when

$$x - \frac{1}{2}\sin(\theta) \geq 0 \text{ and } x + \frac{1}{2}\sin(\theta) \leq 1.$$

That is,

$$x \geq \frac{1}{2}\sin(\theta) \text{ and } x \leq 1 - \frac{1}{2}\sin(\theta).$$

So to compute the area of the large rectangle we have

$$\text{Length x Width} = (\pi/2 - -pi/2) \times 1 = \pi.$$

And the area between the two functions is

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} 1 - \frac{1}{2}\cos(\theta) - \frac{1}{2}\cos(\theta)d\theta &= \int_{-\pi/2}^{\pi/2} 1 - \cos(\theta)d\theta \\ &= \theta - \sin(\theta)|_{-\pi/2}^{\pi/2} \\ &= \pi/2 - \sin(\pi/2) - [-\pi/2 - \sin(-\pi/2)] \\ &= \pi/2 - 1 + \pi/2 - 1 = \pi - 2.\end{aligned}$$

So the probability of the needle *not* touching the line is

$$p_{\text{not touching}} = \frac{\text{area from integral}}{\text{area of rectangle}} = \frac{\pi - 2}{\pi} = 1 - 2\pi.$$

Thus the probability of the needle touching the line is

$$p = 1 - p_{\text{not touching}} = 1 - (1 - 2\pi) = 2\pi.$$

2. Series Approximations

To approximate *pi* with the needle drop is fun, but not as fun as infinite series. So let's look aat a couple of common ones.

$$\frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \tag{1}$$

$$\frac{\pi^2}{6} = \sum_{k=1}^{\infty} \frac{1}{k^2} \tag{2}$$

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103+26390k)}{(k!)^4 396^{4k}} \tag{3}$$

These equations look a bit complicated and we will see why we might want the equation 3 due to Srinivasa Ramanujan. Let's first start with the simplest, equation 1. This equation is easy to work with in a high school since we need only understabnd a littel addition and a little bit of series work. Let's try adding up the first few terms

$$\frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \approx \sum_{k=0}^{5} \frac{(-1)^k}{2k+1}. \tag{4}$$

So
$\frac{\pi}{4} \approx \sum_{k=0}^{5} \frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11}$.
These being easy enough to add up by hand gives us Summed up I

$$\begin{aligned} 1 =& \ 1.0000000\ldots \\ -\tfrac{1}{3} = \text{-}& \ 0.3333333\ldots \\ \tfrac{1}{5} =& \ 0.2000000\ldots \\ -\tfrac{1}{7} = \text{-}& \ 0.1428571\ldots \\ \tfrac{1}{9} =& \ 0.1111111\ldots \\ -\tfrac{1}{11} =\text{-}& \ 0.0909091\ldots \end{aligned}$$

get $\frac{\pi}{4} \approx \sum_{k=0}^{5} \frac{(-1)^k}{2k+1} = 0.7440115\ldots$. So we get $\frac{\pi}{4} \approx 0.7440115\ldots$ and thus $\pi \approx 4 \times 0.7440116 \cdots = 2.9760\ldots$. This technique seems like the needle, it is once again not too close.

What I would like you to do is split into groups and find approximations for π with this series using different values and graph them on with *pi* and try to improve our technique.

pi Approximation with a Series		
Choice of n	Sum of $\sum_{k=0}^{n} \frac{(-1)^k}{2k+1}$	multiple by 4
3		
4		
5	$0.7440116\ldots$	$2.9760464\ldots$
6		
7		
8		

References

[1] Schroeder, L. (1974). *Buffon's needle problem: An exciting application of many mathematical concepts*. Mathematics Teacher, **67** (2), 183–6.

[2] Magdi Ragheb. Monte Carlo Simulation Science and Engineering Applications. http://mragheb.com/NPRE

[3] `http://matwbn.icm.edu.pl/ksiazki/aa/aa27/aa27121.pdf`